Washington Math Olympiad

Year 3

April 7, 2025

Welcome to the solutions packet of the Washington Math Olympiad (WAMO).

1. Compute x given that

$$\sqrt{x} + \sqrt{\sqrt{x}} + \sqrt{\sqrt{\sqrt{x}}} = 93.$$

Ryan Tang

Solution. Let $y = \sqrt{\sqrt{\sqrt{x}}}$. Since $y^4 + y^2 + y = 93$ holds for $y = 3$, we have $x = 3^8 = \boxed{6561}$.

2. A certain school has 120 students. If $X\%$ of these students ride the bus to school, how many distinct integer values of X are possible?

James Yang

Solution. Note that $120 \cdot \frac{X}{100} = \frac{6X}{5}$ must be an integer. Therefore, X must be divisible by 5; there are $\boxed{21}$ such possible nonnegative integer values of X that are less than or equal to 100.

3. Consider a large circle centered at the origin. This circle is partitioned into four quadrants by the x and y axes. Within each quadrant, a circle is inscribed such that it is tangent to both the large circle and the corresponding axes. A fifth circle is externally tangent to all four of these inscribed quadrant-circles. Let the ratio of the area of this fifth circle to the area of the original large circle be R. If R can be expressed as $a - \sqrt{b}$, where a and b are positive integers, compute $a + b$.

Alexander Yang

Solution. The ratio of the areas is the square of the ratio of the radii. Let the radius of the four medium-sized circles to be 1. Then, the radius of the smallest circle is $\sqrt{2} - 1$ and the radius of the largest circle is $\sqrt{2} + 1$. Now calculating, we have $\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^2 = (3 - 2\sqrt{2})^2 = 17 - \sqrt{288}$ giving an answer of $\boxed{305}$.

4. Let $P_1P_2 \ldots P_{20}$ be a regular 20-gon. A subset of the vertices $P_1, P_2, \ldots, P_{20}$ is selected at random, with each non-empty subset of fewer than 20 vertices equally likely. The probability that every element within the chosen subset lies within a circle while all other vertices lie outside of the circle can be expressed as $\frac{m}{n}$, where m and n are positive integers with $\gcd(m, n) = 1$. Compute the sum of the positive factors of m.

Alexander Yang

Solution. Note that every element of a certain subset of points can lie within a circle if and only if the points are consecutive. We must choose a subset of size 1 to 19, yielding $20 \cdot 19 = 380$ total ways to choose the consecutive vertices.

Next, there is a total of $2^{20} - 2$ ways to choose any subset of the vertices, so the probability is just $\frac{380}{2^{20}-2} = \frac{190}{2^{19}-1}$, and the sum of the factors of 190 is $\boxed{360}$.

5. A point M lies inside a circle. Two equilateral triangles $\triangle MPQ$ and $\triangle MRS$ are drawn such that points P, Q, R, S lie on the circle. If MPQ has side length 12, and MRS has side length 18, what is the square of the radius of the circle?

   Wesley Wu

   Solution. Let the center of the circle be O, and let the radius be r. Note that since both equilateral triangles must be symmetric around one diameter, OM must be the altitude of both equilateral triangles. Let the feet of the altitudes from O to PQ and RS be X and Y, respectively. Then, we have $PX = QX = 6$ and $RY = SY = 9$. We also know that $OX + OY = MX + MY = 15\sqrt{3}$.
   Let $OX = a$ and $OY = 15\sqrt{3} - a$. Then, using the Pythagorean theorem on triangles OPX and ORY gives $36 + a^2 = 81 + (15\sqrt{3} - a)^2 = r^2$. Solving, we find that $a = 8\sqrt{3}$. Therefore, the square of the radius is $192 + 36 = \boxed{228}$.

6. Let $a_n = 1^2 + 2^2 + 3^2 + \ldots + n^2$. Compute the number of positive factors of the sum $a_1 + a_2 + a_3 + \ldots + a_{100}$.

   Alexander Yang

   Solution. Note that $k^2 = 2\binom{k}{2} + \binom{k}{1}$, thus by hockey stick $a_n = 1^2 + 2^2 + \cdots + n^2 = 2\binom{n+1}{3} + \binom{n+1}{2}$. By hockey stick once again, $a_1 + a_2 + \cdots + a_{100} = 2\binom{102}{4} + \binom{102}{3} = 2 \cdot 5^2 \cdot 17 \cdot 101^2$, and computing gives $\boxed{36}$.

7. Compute the sum of all possible distinct values of the expression

$$(1 \pm 1)(1 \pm 1 \pm 1 \pm 1)(1 \pm 1 \pm 1 \pm 1 \pm 1),$$

   where each $\pm$ independently represents a choice of $+$ or $-$.

   Ryan Tang

   Solution. The answer is just $(0 + 2)(-2 + 0 + 2 + 4)(-3 - 1 + 1 + 3 + 5) = \boxed{40}$ by considering each part separately.

8. The graph of the equation

$$x^2 + y^2 + 216 = 36(|x| - |y|)$$

   consists of multiple curves. The total length of these curves can be expressed as $a\pi\sqrt{b}$, where a and b are positive integers and b is square-free. Find ab.

   Alexander Yang

Solution. Do casework on each quadrant. Looking at the first quadrant, there is a sixth of a circle with radius $12\sqrt{3}$; this is the same for all other quadrants, leading to an answer of $4 \cdot \frac{1}{6} \cdot 24\pi\sqrt{3} = 16\pi\sqrt{3}$, and computing gives $\boxed{48}$.

9. Let $N = (((1@4)@3)@4)$, where each @ can independently be addition, subtraction, multiplication, or division, each with equal probability. The expected value of N can be expressed as $\frac{m}{n}$, where m and n are positive integers and $\gcd(m, n) = 1$. Compute $m + n$.

   Alexander Yang

   Solution. The expected value of $a@b$ is $\frac{a+b+a-b+a\cdot b+\frac{a}{b}}{4}$; we can just apply this thrice due to the expected values being independent. Computing yields $\frac{625}{192}$, from which we extract our final answer of $\boxed{817}$.

10. A 11-digit arrangement of the set $\{0, 1, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is called *amazing* if

    - 0, 1, 3, 4 appear in increasing order (not necessarily consecutively), and
    - 5, 6, 8, 9 appear in decreasing order (not necessarily consecutively).

    For example, $9, 1, 8, 0, 1, 6, 7, 3, 2, 4, 5$ and $2, 9, 8, 0, 1, 6, 7, 3, 1, 4, 5$ are *amazing* arrangements. How many *amazing* arrangements are there?

    Bruce Shu

    Solution. We do casework on where the second 1 will go. It can either be in front of the 0, in between 0 and 3, in between 3 and 4, or after 4. For each of these 4 cases, there are $\binom{11}{5} \cdot \binom{6}{2} \cdot 2!$ ways to arrange the numbers. Thus, our answer is $\boxed{55440}$.

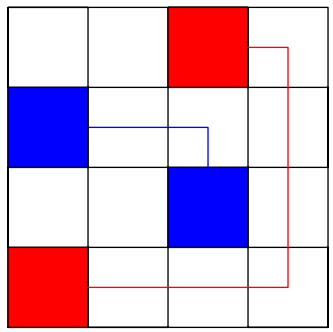
11. Let $a, b, c,$ be positive integers such that the expression $(a - \frac{1}{b})(b - \frac{1}{c})(c - \frac{1}{a})$ is also a positive integer. Compute the positive difference between the maximum and minimum possible values of the expression $7a + 8b + 9c$.

    Kaiyuan Mao

    Solution. Expanding the expression, we get $abc - a - b - c + \frac{ab+bc+ca-1}{abc}$. Without loss of generality, let $a \leq b \leq c$. Then, $abc | ab + bc + ca - 1$, so $0 < abc \leq ab + bc + ca - 1 < 3bc$. Thus, $a < 3$, meaning that $a = 1$ or 2.
    If $a = 1$, then $bc | bc + b + c - 1$, or $bc | b + c - 1$. Then, $bc - b - c \leq -1$, and $(b - 1)(c - 1) \leq 0$, so $b = c = 1$. However, this makes the expression equal to zero, which is not a positive integer.
    If $a = 2$, then $2bc | bc + 2(b + c) - 1$. We have $2bc \leq bc + 2(b + c) - 1$, or $(b - 2)(c - 2) \leq 3$. Thus, we obtain $b = 2$ or $b = 3$ and $3 \leq c \leq 5$ (since $a \leq b \leq c$). Testing out $b = 2$ leads to $4c | 4c + 3$, which is never satisfied. Testing out $b = 3$ leads to $6c | 5c + 5$, which is only satisfied when $c = 5$. Thus, our only ordered triple is $(2, 3, 5)$.

We can restore generality now. By the rearrangement inequality, the maximum of $7a+8b+9c$ occurs at $(2,3,5)$, while the minimum occurs at $(5,3,2)$. Calculating, our answer is $\boxed{6}$.

12. In a 4×4 grid of unit squares, two squares are colored red and two are colored blue. A configuration is called *connected* if both red squares are connected by a red path passing through the edges of the grid, and the blue squares are connected by a similar blue path. The red and blue paths, including the start and end squares, must not intersect. How many configurations are *not connected*?



Wesley Wu

Solution. Denote red squares with A and blue squares with B. There are three configurations that are not connected. The first is when there is a 2 by 2 block with top row AB and bottom row BA. The second is when all four marked squares are on the edge and the A's and B's alternate when read clockwise. The last is when there is an A or a B in one corner and the two adjacent cells are filled with the opposite color. Note that the first two cases do not suffer from overcounting.

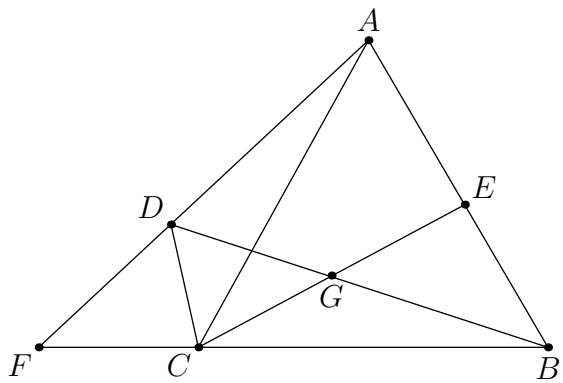
For the first case, the block has two possible configurations per 2 by 2 block–when A is in the top left corner or when B is there. There are 9 spots to place the block, so the total for this case is 18.

For the alternating case, there are 12 edge squares, of which we must select 4. Afterwards, we must assign an order–consider the first marked square moving clockwise from the top left corner. We can choose for that square to be A or B. Our count for this case is $\binom{12}{4} \cdot 2 = 990$.

Finally, we have the "trapped" case. Note that to avoid overcounting, the last marked square cannot be on the edges or in the square adjacent to both colors, as these would change the configuration into one of the aforementioned two. Thus, we pick a corner, a color, and the fourth square (of which we have three remaining options) to get $4 \cdot 2 \cdot 3 = 24$.

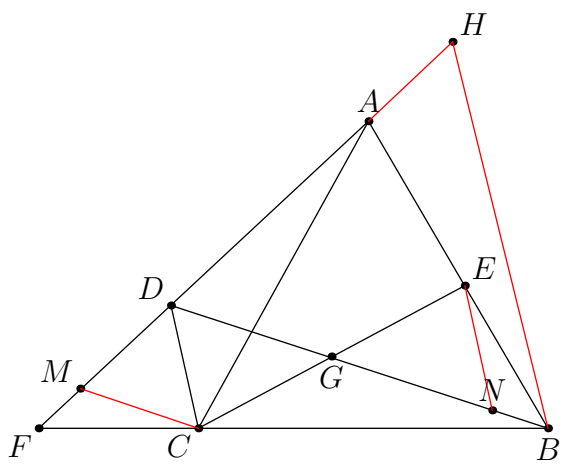
Summing these all together, our final answer is $\boxed{1032}$.

13. As shown in the diagram below, D is a point outside equilateral triangle ABC with $\angle ADC = 120°$. Lines AD and BC are extended to meet at point F, and point E lies on segment AB such that $BE = CF$. Given that $DG = 14$ and $AF = 34$, compute the length of segment BD.



Kaiyuan Mao

Solution.



Since $\angle ADC + \angle ABC = 180°$, quadrilateral $ABCD$ is cyclic. This yields $\angle ADB = \angle ACB = 60°$ and $\angle CDB = \angle CAB = 60°$, while $\angle DCB + \angle DAB = 180°$. By Ptolemy's, $BD = AD + DC$, which inspires us to rotate $\triangle CDB$.

Rotating $\triangle CDB$ $60°$ clockwise about B maps it to $\triangle AHB$, as $CB = AB$ and $\angle ABC = 60°$. From $\angle DCB + \angle DAB = 180°$, H, A, and D are collinear and $\triangle HDB$ is equilateral.

Construct M on segment DF such that $\triangle DMC$ is equilateral. Then, $DM = DC$, and since $AD + DM = AM$, we find that $BD = AM = AF - FM$. Let $FM = x$, causing $BD = 34 - x$.

Now, construct point N on BG such that $BN = FM = x$, and let $\angle NBE = \alpha$. Because $ABCD$ is cyclic, $\angle ACD = \alpha$. The exterior angle theorem gives $\angle CMF + \angle MFC = \angle MCB = \angle MCD + \angle DCA + \angle ACB = 120^\circ + \alpha$, so $\angle MFC = \alpha = \angle NBE$.

Since we know that $BN = FM$, $\angle MFC = \angle NBE$, and $FC = BE$, we can find that $\triangle MFC \cong \triangle NBE$. Thus, $NE = MC = DC$, and $\angle GNE = 180^\circ - \angle ENB = 60^\circ = \angle GDC$.

By AAS, $\triangle GNE \cong \triangle GDC$, so $NG = DG = 14$ and $BD = DG + NG + BN = 28 + x$. Equating with the earlier expression $BD = 34 - x$ gives $34 - x = 28 + x$, so $x = 3$ and $BD = \boxed{31}$.

14. The value of

$$\sum_{k=2}^{9999} \frac{101 - \lceil \sqrt{k+2} \rceil}{k(k+1)}$$

can be expressed as $\frac{m}{n}$, where $\gcd(m, n) = 1$. Compute the remainder when $m+n$ is divided by 1000.

Ryan Tang

Solution. The denominator instantly reminds us of telescoping, so we can rewrite the expression as $\sum_{k=2}^{9999} \left(101 - \lceil \sqrt{k+2} \rceil\right) \left(\frac{1}{k} - \frac{1}{k+1}\right)$. Observe that $101 - \lceil \sqrt{k+2} \rceil$ stays constant as k changes from $a^2 - 1$ to $(a+1)^2 - 2$, inclusive. Then, $\frac{1}{k} - \frac{1}{k+1}$ telescopes, and every term cancels out except for $\frac{1}{a^2-1}$ and $\frac{1}{(a+1)^2-2+1}$, so our expression simplifies to $\frac{99}{2} - \frac{99}{3} + \sum_{a=2}^{99} (100 - a) \left(\frac{1}{a^2-1} - \frac{1}{a^2+2a}\right)$. Since $\frac{1}{(a+1)^2-1} = \frac{1}{a^2+2a}$ (which can be found by factoring the denominator), we can pair $-\frac{100-a}{a^2+2a}$ and $\frac{100-(a+1)}{(a+1)^2-1}$ to reduce the expression to

$$\frac{33}{2} + \frac{98}{3} - \frac{1}{99 \cdot 101} - \sum_{a=2}^{98} \frac{1}{a^2 + 2a} = \frac{33}{2} + \frac{98}{3} - \frac{1}{99 \cdot 101} - \frac{1}{2} \sum_{a=2}^{98} \left(\frac{1}{a} - \frac{1}{a+2}\right).$$

Again, our summation telescopes, so we can find the closed form

$$\frac{33}{2} + \frac{98}{3} - \frac{1}{99 \cdot 101} - \frac{1}{2} \left(\frac{1}{2} + \frac{1}{3} - \frac{1}{99} - \frac{1}{100}\right),$$

which, after some bashing, evaluates to $\frac{984951}{20200}$, which gives us the answer $\boxed{1005151}$.

15. Let (x, y) be an ordered pair of real numbers satisfying the system of equations

$$\sqrt{x^2 + y^2 - 8x + 8y + 32} + \sqrt{x^2 + y^2 + 8x - 8y + 32} = 10\sqrt{2}$$
$$5(xy + 8)(x^2 - y^2) = 1224$$

Compute the sum of all distinct possible values of $x^2 + y^2$.

Wesley Wu

Solution. Note that the first equation can be rewritten as

$$\sqrt{(x-4)^2 + (y+4)^2} + \sqrt{(x+4)^2 + (y-4)^2} = 10\sqrt{2}.$$

This is exactly of the form

$$\text{distance from } (x, y) \text{ to } F_1 + \text{distance from } (x, y) \text{ to } F_2 = \text{constant},$$

with $F_1 = (4, -4)$ and $F_2 = (-4, 4)$. Thus the set of all points (x, y) satisfying the equation is an ellipse. Specifically, the ellipse in this definition has semimajor axis $5\sqrt{2}$, semiminor axis $3\sqrt{2}$, and focal distance $4\sqrt{2}$.

However, it is necessary to note that the ellipse, as given, is "tilted" with respect to the x-y axes, meaning that a rotation simplifies the ellipse to standard form. The foci $(4, -4)$ and $(-4, 4)$ lie along a line at a $45°$ angle from the axes, so rotating the coordinate system by $45°$ makes the axes of the ellipse align with the new coordinate directions.

A $45°$ rotation is performed using the transformation:

$$X = \frac{x-y}{\sqrt{2}}, \ Y = \frac{x+y}{\sqrt{2}}.$$

(Notice that when applying a rotation to the entire graph, we must be careful with the direction so that the new coordinates describe points on the transformed ellipse.)

Under this rotation, the ellipse can be expressed in standard form as

$$\frac{X^2}{18} + \frac{Y^2}{50} = 1.$$

Thus, a convenient parametrization of the rotated ellipse is:

$$X = 3\sqrt{2}\cos\theta, \ Y = 5\sqrt{2}\sin\theta.$$

Afterwards, we may interpret the relation for x^2+y^2 in terms of θ once we "rotate back" to the original coordinates.

Now, since our ellipse has been "untwisted" by the rotation, it is helpful to transform the second equation accordingly. This changes our second equation into

$$5XY(Y^2 - X^2 - 16) = 1224.$$

Substituting X and Y in terms of θ reduces the equation to

$$\frac{1224}{5} = 30\sin\theta\cos\theta \cdot (50\cos^2\theta - 18\sin^2\theta - 16).$$

Using the double and half-angle formulas

$$\sin(2\theta) = 2\sin\theta\cos\theta,\ \ \cos^2\theta = \frac{1+\cos(2\theta)}{2},\ \ \sin^2\theta = \frac{1-\cos(2\theta)}{2},$$

the equation simplifies to an expression involving $\sin(4\theta)$:

$$255\sin(4\theta) = \frac{1224}{5}.$$

Thus,

$$\sin(4\theta) = -\frac{24}{25}.$$

After the rotation and parametrization, one can "trace back" the expression for the sum of squares. In the rotated coordinate frame the point has coordinates related to θ; translating this back leads to

$$x^2 + y^2 = 18\cos^2\theta + 50\sin^2\theta.$$

This expression can be rewritten as

$$x^2 + y^2 = 18 + 32\sin^2\theta.$$

Now we attempt to identify the possible values of $\sin^2\theta$.

Using half-angle formulas yield that the possible values of $\sin^2\theta$ are

$$\frac{1}{5},\ \frac{4}{5},\ \frac{1}{10},\ \frac{9}{10}.$$

Doing some final calculations yields an answer of $2\cdot(2\cdot 18+32) = \boxed{136}$.

The 2025 Washington Math Olympiad 3 was made possible by the contributions of the following problem writers and test solvers:

Ryan Chen	(Eastlake)
Jason Lee	(NC)
Kaiyuan Mao	(Odle)
Bruce Shu	(TX)
Patrick Sun	(Interlake)
Ryan Tang	(Newport)
Laura Wang	(Lakeside)
Wesley Wu	(Interlake)
Alex Yang	(Odle)
James Yang	(Lakeside)